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Sustainability as Intergenerational Fairness: Efficiency, Uncertainty, and Numerical Methods

Richard T. Woodward

This paper presents an economic model of *sustainability* defined as intergenerational fairness. Assuming that intergenerational fairness is an obligation of each generation, a recursive optimization problem is obtained. The problem has the advantage that uncertainty can readily be incorporated in the model and it can be solved numerically for a wide range of specifications. The possibility of trade-offs between efficiency and sustainability are discussed. Under plausible conditions, it is shown that a sustainability obligation is met only if there is the expectation of economic growth.

Key words: dynamic programming, intergenerational equity, numerical methods, sustainability, uncertainty.

Throughout the history of economic thought there has been enormous interest in the issue of intergenerational equity (Ramsey). In recent years, most of this discussion has taken place using the terms *sustainable development* or *sustainability* and has paid particular attention to the role of natural resources and the environment in sustaining economic wellbeing. Both optimal control (e.g., Solow 1974) and overlapping generations (e.g., Howarth 1991) models have been used to address whether a sustainable economy is feasible and efficient. An excellent review of this literature is provided by Toman, Pezzey, and Krautkraemer.

The present paper contributes to this literature in three ways. First, we incorporate into our model an interpretation of sustainability based on Foley's principle of fairness—a generation is defined as behaving sustainably if it does not expect to be envied by future generations. Given Rawls' characterization of justice as fairness, this approach is conceptually quite similar to a Rawlsian maximin objective that has predominated in the economics literature.[1] Following Riley, and similar to writings by political philosophers (Laslett and Fishkin), sustainability is treated as an obligation of the current generation to future generations. Since the welfare of each generation is assumed to be altruistic, we obtain results similar to those of Calvo.

Second, we explicitly incorporate risk into the analysis. Ironically, although long-term uncertainty is one of the central issues associated with sustainability, it has received scant treatment in the economics literature. Notable exceptions are Howarth (1995, 1997), Toman, and Asheim and Brekke. While Howarth (1995) suggests the use a sustainability criterion equivalent to the one we propose below, both he and Toman end up appealing to rules of thumb for policy formation to achieve sustainability. Although such rules may be useful in practice, in this paper we seek to understand their theoretical underpinnings so as to better evaluate their ability to achieve the goals of efficiency and sustainability. Our framework has many similarities to Asheim and Brekke, though they use a non-altruistic framework.

Finally, we show that numerical methods can be used to find optimal-sustainable policies. This represents an important contribution in that it substantially broadens

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[1] The problem of intergenerational equity has been studied using a maximin objective at least since Solow (1974) and continues to this day (Pezzey and Withagen).

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the scope of the problems that economists can consider. Analytical methods are limited because closed-form solutions can be obtained only for relatively simple problems. The introduction of numerical methods, therefore, can broaden the spectrum of problems that might be considered. These methods are then used to evaluate sustainability in simple one- and two-dimensional economies.

The Model

We assume that the welfare of generation t can be written as an additive function of the infinite stream of utility of all future generations, i.e.,

$$(1) \quad U({}_t\mathbf{z}, {}_t\mathbf{x}, {}_t\varepsilon) = E_t \sum_{s=t}^{\infty} \beta^{s-t} u(\mathbf{z}_s, \mathbf{x}_s, \varepsilon_s)$$

where $\mathbf{x}_t \in X$ is a vector describing generation t's endowment, $\mathbf{z}_t$ is the vector of choices made by the tth generation, and $\varepsilon_t \in \Lambda$ is the vector of identically independently distributed (i.i.d.) stochastic shocks that occur after the choices, $\mathbf{z}_t$, have been made. The notation ${}_t\mathbf{z}$ indicates the infinite series of choices, $\mathbf{z}_t, \mathbf{z}_{t+1}, \mathbf{z}_{t+2}, \dots$, and likewise for ${}_t\mathbf{x}$ and ${}_t\varepsilon$. The parameter $\beta < 1$ is the societal discount factor[2] and E_t is the expectation operator contingent on the information available to generation t as captured in the endowment vector $\mathbf{x}_t$. The functions $u(\cdot)$ and $U(\cdot)$ will be called the generational utility and welfare functions, respectively. We assume that there is a set of m stationary transition functions, $x^i_{t+1} = g^i(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t), i = 1, \dots, m$, which we will frequently write in vector notation

$$(2) \quad \mathbf{x}_{t+1} = g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t).$$

The set of feasible choices depends on the state of the economy, $\Gamma(\mathbf{x}_t) \subset \mathbb{R}^n$.

Assuming each generation chooses to maximize its welfare, (1) can be rewritten recursively,

$$(3) \quad V(\mathbf{x}_t) = \max_{\mathbf{z}_t \in \Gamma(\mathbf{x}_t)} E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \beta E_t V(\mathbf{x}_{t+1}) \qquad \text{s.t}(2).$$

[2] Pezzey (1997) explores conditions where the discount factor may vary over time, an innovative approach that merits further investigation.

Sustainability as Intergenerational Fairness

Most of the economic analysis concerned with *sustainability* has taken its motivation from Rawls' principle of justice.[3] Accordingly, economists have evaluated numerous problems in which the objective is to maximize the minimum utility across all future generations (e.g., Solow 1974, Hartwick 1977). Calvo modified the model by assuming that welfare is equal to the discounted sum of future utility (see also Asheim 1988 and Rodriguez) leading to optimal maximin plans that can be dynamically consistent and can lead to growth over time.

In the present paper we use a slightly different approach, basing our analysis on the principle of intergenerational fairness. Foley defines an allocation as fair, "if and only if each person in the society prefers his [or her] consumption bundle to the consumption bundle of every other person in the society" (p. 74). The allocation of a multidimensional endowment $\mathbf{x}$ between n individuals is fair if each individual prefers his or her bundle to that of every other individual, i.e., $u_i(\mathbf{x}_i) \geq u_i(\mathbf{x}_j)$ for all i, j.[4]

Following Foley, we define choices by the current generation as *intergenerationally unfair* if, given $\mathbf{z}_t$, either the current generation envies future generations or future generations envy the present, and there exists an alternative feasible choice such that there is no envy. We define a set of choices, $\mathbf{z}_t$, as *sustainable* or *consistent with sustainability* if they are intergenerationally fair.

A set of choices, $\mathbf{z}_t$, is fair to future generations if

$$\mathrm{FF}_t\colon E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \beta E_t U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_t} \leq E_t U({}_{t+j}\mathbf{z}, {}_{t+j}\mathbf{x}, {}_{t+j}\varepsilon)\big|_{\mathbf{z}_t} \quad \text{for all } j = 1, 2, \dots .$$[5]

To take the expectation in FF_t, generation t must make some assumptions about how

[3] Chichilnisky provides an important alternative approach to the issue of sustainability. Using axioms that rule out dictatorship of either the present or the indefinite future, she derives an objective function made up of the weighted sum of a present-value function and the least utility of the distant future. Neither a Rawlsian objective function nor the sustainability-constrained objective put forth here satisfy these axioms.

[4] Following Foley's original definition, a number of similar definitions were proposed by economists during the 1970s (see Thomson and Varian for a summary).

[5] This definition was also considered by Howarth (1995) and is the discrete-time and stochastic analogue of Riley's definition.

future generations will make choices. For example, the endowment of generation $t+2$ depends not only on the choices and endowment of generation t, but also on the choices of generation $t+1$. The assumption regarding future choices would typically take the form of a *policy rule*, a mapping from the endowment vector $\mathbf{x}$ to a choice vector $\mathbf{z}$. We make the following assumption about the policy rule of future generations.

ASSUMPTION A. FF_s *is satisfied for all future generations,* $s = t+1, t+2, \ldots$.

If Assumption A does not hold, then virtually any effort to treat distant generations fairly might be undone by the unfair choices of intervening generations and choices made in t would be dynamically inconsistent. If Assumption A does hold then sustainability is achieved if each generation is incrementally fair to the following generation.

PROPOSITION 1. FF_t *is satisfied for all* t *if and only if*

$$\begin{aligned} IFF_t\colon\ & E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \\ & \beta E_t U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_t} \\ & \le E_t U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_t} \end{aligned}$$

is satisfied for all t.

Proof. FF $\Rightarrow$ IFF: This is automatic since FF_t implies IFF_t.

IFF $\Rightarrow$ FF: IFF_{t+1} implies that

$$\begin{aligned} & E_{t+1} U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_t+1} \\ & \le E_{t+1} U({}_{t+2}\mathbf{z}, {}_{t+2}\mathbf{x}, {}_{t+2}\varepsilon)\big|_{\mathbf{z}_{t+1}}. \end{aligned}$$

Taking expectations based on the information available to generation t and contingent on the decision $\mathbf{z}_t$, $E_t[E_{t+1}U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_{t+1}}]\big|_{\mathbf{z}_t} \le E_t[E_{t+1}U({}_{t+2}\mathbf{z}, {}_{t+2}\mathbf{x}, {}_{t+2}\varepsilon)\big|_{\mathbf{z}_{t+1}}]\big|_{\mathbf{z}_t}$. Hence, using the law of iterated expectations, IFF_t and IFF_{t+1} together imply

$$\begin{aligned} & E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \\ & \beta E_t U({}_{t+1}\mathbf{z}, {}_{t+1}\mathbf{x}, {}_{t+1}\varepsilon)\big|_{\mathbf{z}_t} \\ & \le E_t U({}_{t+2}\mathbf{z}, {}_{t+2}\mathbf{x}, {}_{t+2}\varepsilon)\big|_{\mathbf{z}_t}. \end{aligned}$$

By induction, the same relationship holds replacing $t+2$ with $t+3$ and so on.

For non-altruistic preferences, $\beta = 0$, Proposition 1 is equivalent to Asheim and Brekke's Lemma 3. This proposition leads to what Howarth (1992) refers to as a "chain of obligation" in which an obligation to treat our immediate descendants fairly implies an obligation to all future generations. But Proposition 1 is more general—an obligation of fairness to all future generations is satisfied if each generation is suitably fair to the following generation.

As we have defined it, sustainability is a symmetric criterion, implying a lack of envy by future generations of the present and vice versa. The problem of intergenerational choice, however, is fundamentally asymmetric. While the present is able to act without regard to the interests of the future, the future has no choice but to accept the actions of the present (Bromley). If the present generation is acting in its best interest it will maximize its welfare, minimizing the extent to which it envies future generations. This temporal advantage ensures that, to the extent possible, the present should not envy the future.

The temporal disadvantage of the future, on the other hand, means that there is no guarantee that optimal behavior will lead to choices that are fair to future generations. Laslett maintains that the rights of earlier generations must be matched by duties to generations yet to come. To ensure that sustainability is achieved, therefore, the optimization problem that the current generation solves must be altered.

Sustainability-Constrained Optimization

Intergenerational fairness requires that, to the extent possible, the current generation is both fair to the future and to itself. This is achieved if all generations solve the following sustainability-constrained optimization problem:

$$\begin{aligned} V^S(\mathbf{x}_t) = \max_{\mathbf{z}_t} & E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\ & + \beta E_t V^S(\mathbf{x}_{t+1}) \qquad \text{s.t.} \end{aligned}$$

$$\begin{aligned} (4) \qquad & \text{(i) } \mathbf{z}_t \in \Gamma(\mathbf{x}_t) \\ & \text{(ii) } \mathbf{x}_{t+1} = g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\ & \text{(iii) } V^S(\mathbf{x}_t) \le E_t V^S(\mathbf{x}_{t+1}). \end{aligned}$$

The first and second constraints identify intra- and intertemporal feasibility, respectively. The third constraint is the sustainability constraint. We denote the value function of this problem $V^S(\mathbf{x})$ to differentiate it from $V(\mathbf{x})$ defined in (3).

The existence of a solution to (4) is ensured by an assumption of free disposal. Let $\underline{u}$ be the lowest possible level of utility, i.e., $\underline{u} = \inf_{\mathbf{z},\mathbf{x},\varepsilon} u(\mathbf{z}, \mathbf{x}, \varepsilon)$.

ASSUMPTION B. *For all* $\mathbf{x} \in X$ *there exists some* $\underline{\mathbf{z}} \in \Gamma(\mathbf{x})$ *such that* $u(\underline{\mathbf{z}}, \mathbf{x}, \varepsilon) = \underline{u}$ *for all* $\varepsilon \in \Lambda$.

Assumption B simply means that any generation can always reach $\underline{u}$ by acting in a suitably wasteful manner. It can easily be shown (see Lemma 3 in the Appendix) that choosing $\underline{\mathbf{z}}$ forever satisfies the three constraints of (4).

Incorporating sustainability as a constraint may be disconcerting to some readers. It might be argued that if society *chooses* to constrain itself then there must be some other, more fundamental, objective function that it is truly maximizing.[6] Dasgupta and Mäler argue that this underlying function should be modeled to allow the consideration of trade-offs between sustainability and other goals. Sen provides two reasons why explicit analysis of the constraint might be useful. First, he shows that modeling self-imposed constraints directly can be functionally important. Second,

> Even if it were the case that—"ultimately"—everything were determined by "basic" preferences exclusively over *culmination outcomes*, it would still be interesting and important to see how the derived preferences ("nonbasic" but functionally important) actually work in relation to the choice act (p. 749, emphasis in original).

Hence, modeling sustainability as a constraint can yield important insights into how policy might evolve *if* intergenerational fairness is held as a generational obligation.[7]

The Solution of Sustainability-Constrained Optimization Problems

Having proposed (4) as an interesting social optimization problem, we now need to consider whether such a problem can actually be solved. This is not a trivial question. The sustainability-constrained value function $V^S(\mathbf{x})$ appears four times in the statement of the problem, on both the right- and left-hand sides of the objective function and twice again in the sustainability constraint. Yet this function, like the unconstrained value function in standard dynamic programming problems, is unknown a priori. Except in very restrictive cases, it is impossible to solve (4) analytically and typically there will be no closed form for $V^S(\cdot)$.[8] Fortunately, a unique sustainability-constrained value function exists for a wide variety of economies and can be found using numerical methods.

The algorithm that can be used to solve sustainability-constrained optimization problems proceeds in two basic steps. The first step is to solve the unconstrained optimization problem, (3), yielding the unconstrained value function $V^*(\mathbf{x})$.[9]

Then, using this value function as our initial guess, $V^0 = V^*$, the sustainability-constrained value function $V^S(\mathbf{x})$ is found by recursively solving the problem

$$
\begin{aligned}
V^k(\mathbf{x}_t) = \max_{\mathbf{z}_t} \, & Eu(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\
& + \beta E V^{k-1}(\mathbf{x}_{t+1}) \qquad \text{s.t.} \\
\text{(5)} \qquad & \text{(i) } \mathbf{z}_t \in \Gamma(\mathbf{x}_t) \\
& \text{(ii) } \mathbf{x}_{t+1} = g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\
& \text{(iii) } Eu(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \beta E V^{k-1}(\mathbf{x}_{t+1}) \\
& \qquad \le E V^{k-1}(\mathbf{x}_{t+1})
\end{aligned}
$$

at each point in the state space. Proposition 4 in the Appendix proves that this algorithm leads to the unique sustainability-constrained value function V^S, as $k \to \infty$.

While numerical methods can be used to solve many problems, they do face some notable limitations. Most importantly, since computers must operate in finite space, only problems that are bounded can be solved using these methods.

ASSUMPTION C. *The sets* X, Λ, *and* $\Gamma(\mathbf{x})$ *are all closed and bounded and the instantaneous utility function* $u(\mathbf{x}, \mathbf{z}, \varepsilon)$ *is defined over the entire domain.*

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[6] While many economists view a constraint-based approach with suspicion, one referee has pointed out that using a social welfare maximization approach to address principles of fairness is "out of fashion in moral philosophy and political theory." In that literature the constraint-based approach is quite well developed.

[7] A generation may choose to violate this self-imposed constraint if it conflicts with other objectives such as Pareto efficiency. We discuss the possibility of such a conflict below.

[8] Asheim and Brekke present two simple stochastic models for which analytical solutions are possible. The deterministic capital-resource economy of Dasgupta and Heal can also be solved analytically (e.g., Stiglitz, Pezzey and Withagen).

[9] This problem could be solved by a number of methods (see Judd).

This assumption is not met in all economies of interest. For example, it rules out the use of a logarithmic utility function defined over the non-negative real numbers and problems in which the solution requires unbounded accumulation of one or more assets as in Dasgupta and Heal. Nonetheless, some unbounded problems can be solved approximately by innocuously modifying state space and/or utility function.

Sustainability under Uncertainty

It is worth pausing momentarily to consider the role of uncertainty in our sustainability-constrained model. Two conceptual complications arise in applying the expectation operator to the problem of intergenerational fairness. First, $\mathbf{z}_t$ and ε_t determine not only the endowment in $t+1$ but, in effect, the very generation that arrives in $t+1$. Howarth (1995, p. 422) argues, therefore, that the expectation operator is adding across "logically distinct state-contingent generations." If this is true and sustainability requires the current generation to be fair to any possible future generation, then a sustainability criterion must hold not on average but for all possible values of ε. A disadvantage of such a criterion is that it could be impossible to achieve or could require choices that would in most states leave both current and future generations worse off than without a sustainability obligation. Moreover, a requirement that $V(\mathbf{x}_t) \leq V(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t))$ for all possible ε_t would likely be unfair to the current generation. Such a constraint would almost certainly lead to a situation in which the current generation envies the future.

The second complication that arises in the treatment of uncertainty is the applicability of the subjective expected utility (SEU) framework to represent the preferences of each generation. There is an enormous body of evidence suggesting that individuals frequently violate the axioms of the SEU hypothesis (see Machina). This is particularly true when faced with highly uncertain problems or problems in which there are highly negative consequences with small probabilities. The normative foundation for using a SEU framework for policy choice in such problems has been questioned (Asheim and Brekke, Manski, Woodward and Bishop). While these problems merit further discussion, we retain the SEU framework here on the belief that improved understanding of sustainability will be achieved incrementally.

There is an immediate implication for the time-path of welfare for a sustainability-constrained economy under risk. Consider the case where x is one dimensional. The sustainability constraint requires that $E_t V^S(\mathbf{x}_{t+1}) \geq V^S(\mathbf{x}_t)$. If $V^S(\cdot)$ is monotonically increasing and strictly concave, then, by Jensen's inequality, sustainability is satisfied only if the current generation expects the endowment to grow.

The extension of this result to the case of a multiple-dimensional state space requires more careful specification of what is meant by growth. Let the endowment vector be composed of m elements, $x_t^1, \dots x_t^m$, each of which is defined so that in the neighborhood around the actual current endowment, $\mathbf{x}_t$, sustainable welfare is increasing in x_t^i for all i. Taking a first order approximation of the change in $V^S(\cdot)$ with respect to a change from $\mathbf{x}_t$ to $\mathbf{x}_{t+1}$ yields $\Delta V^S \approx \sum_i \partial V^S(\mathbf{x}_t)/\partial x^i \cdot (x_{t+1}^i - x_t^i)$. Letting $\partial V^S/\partial x^1$ be a numeraire, the approximate change in the value of the endowment can be written $\sum_i \tilde{p}_t^i \cdot (x_t^i - x_{t+1}^i)$, where $\tilde{p}_t^i = (\partial V^S/\partial x^i)/(\partial V^S/\partial x^1)$ and $\tilde{p}_t^1 = 1$. Generation t's expectation of the change in the value of the endowment can therefore be approximated as

$$(6) \quad \frac{E_t \Delta V^S}{\partial V^S(\mathbf{x}_t)/\partial x^1} \approx \int_\varepsilon \Big[\sum_i \tilde{p}_t^i \cdot [x_t^i - g^i(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)] \Big] \times h(\varepsilon_t; \mathbf{x}_t) d\varepsilon_t$$

where $h(\cdot)$ is the joint probability density function of ε_t conditional on $\mathbf{x}_t$. However, if $V^S(\cdot)$ is strictly concave the sign of the error due to the linear approximation in (6) is negative. Hence, sustainability is *not* satisfied if generation t expects that, on average, the value of generation $t+1$'s endowment will be just equal to that of its own endowment.

There has been substantial attention of the use of the framework of national accounting to evaluate the sustainability of economies (e.g., Hartwick 1977, 1990; Solow 1993). The basic lesson of this literature [under very restrictive conditions regarding population, technology, and preferences and with an important caveat due to Asheim (1994) and Pezzey (1994)[10]] has been that the mainte-

[10] Asheim (1994) and Pezzey (1994) show that the market value of an economy's assets can be growing over a period during which the consumption is at a level that cannot be sustained indefinitely.

nance of the value of an economy's endowment is an indicator of sustainability. We find here a new implication for planning: under risk, sustainability can typically be achieved only if planners aim not for the maintenance of the economy's value, but for its growth.

Sustainability-Constrained Optimization and Pareto Efficiency

Having proposed an alternative to the standard present-value optimization problem, we now consider the compatibility of this norm with Pareto efficiency. Let $\mathbf{z}_t^S$ be the optimal-sustainable policy for generation t. One way this policy might be inefficient is if there is an alternative policy, $\mathbf{z}_t$, that would either increase the welfare of generation t, or that of generation $t+1$, without diminishing the welfare of the other. That is, $\mathbf{z}_t$ is *stepwise Pareto superior* to $\mathbf{z}_t^S$ if

$$\text{(7a)} \quad E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) + \beta E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)) \geq E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t) + \beta E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$$

and

$$\text{(7b)} \quad E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)) \geq E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$$

with a strict inequality holding in at least one case. If such an inefficiency were identified, it seems likely that policy makers would want to reconsider their commitment to sustainability.

PROPOSITION 2. *Assuming that the solution to the sustainability-constrained optimization problem (4), $(\mathbf{z}_t^S, \lambda)$, is a saddle point,*[11] *then, if $\lambda > 1$, this solution is not efficient and, if $\lambda < 1$, the solution is stepwise efficient.*

The proof to Proposition 2 is provided in the Appendix. Stepwise inefficiencies are particularly common when $V^S(\mathbf{x})$ is bounded. For example, consider an infinite-horizon cake-eating economy in which z represents consumption of a single nonrenewable stock x, so that $x_{t+1} - x_t = z$, and with $u(z = 0) = 0$ and $u' > 0$. Since no finite level of consumption can be sustained indefinitely, the sustainability-constrained optimal level of consumption is zero for all generations so that $V^S(x) = 0$ for all $x \geq 0$. A Pareto improvement over this policy would be for any generation to consume a portion of the stock. But if $z_t > 0$, then $u(z_t) + \beta V^S(x_{t+1}) > V^S(x_{t+1})$, violating the sustainability constraint.

In a diverse economy, however, it is quite unlikely that stepwise inefficiencies will arise. Proposition 3 (below) shows that if there exists a means by which the welfare of all future generations can be increased, then the sustainability-constrained optimum cannot be stepwise inefficient. A *sustainably-productive alternative* to a choice $\mathbf{z}_t$, given the endowment $\mathbf{x}_t$, is an alternative $\mathbf{z}_t' \in \Gamma(\mathbf{x}_t)$, such that for some $\delta > 0$

$$E_t U\left({}_{t+j}\mathbf{z}, {}_{t+j}\mathbf{x}, {}_{t+j}\varepsilon\right)\big|_{\mathbf{z}_t'} \geq E_t U\left({}_{t+j}\mathbf{z}, {}_{t+j}\mathbf{x}, {}_{t+j}\varepsilon\right)\big|_{\mathbf{z}_t} + \delta \quad \text{for all } j = 1, 2, \ldots$$

where the expectation operator on both sides is based on the same policy rule. It is not uncommon for sustainably productive alternatives to exist. For example, consider an economy in which utility is solely a function of the amount of corn consumed, z_t, from the current stock and next period's seed stock is defined by the equation $x_{t+1} = rx_t - z_t$, where $r > 1$ is the crop's rate of growth. In such an economy, consumption in all future periods can be increased by $(r-1)\Delta z_t$ simply by reducing consumption today by Δz_t.

ASSUMPTION D. *Utility can be discarded so that for any combination of $\mathbf{z}_t$, $\mathbf{x}_t$, and ε_t, and for any $\alpha \in [\underline{u}, u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)]$, there exists an alternative choice $\mathbf{z}_t'$ such that $u(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t) = \alpha$ and $g^i(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t) = g^i(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)$ for all i.*

PROPOSITION 3. *Suppose Assumption D holds and there exist sustainably productive alternatives for all $\mathbf{z}_t \in \Gamma(\mathbf{x}_t)$. If $\mathbf{z}_t^S$ is the sustainability-constrained optimal choice at $\mathbf{x}_t$, then it is stepwise Pareto efficient.*

The proof is provided in the Appendix. Proposition 3 shows that the potential for stepwise inefficiencies is quite limited. If there exists a way to increase the welfare of all future generations, regardless of the relative cost to the current generation, then this type of inefficiency is avoided. Hence, if there is an asset in the economy like corn in the example above, then this conflict between efficiency and sustainability cannot arise.

[11] This restriction is satisfied in many, but by no means all, problems of interest. See Takayama for a discussion of conditions when a saddle point is not guaranteed.

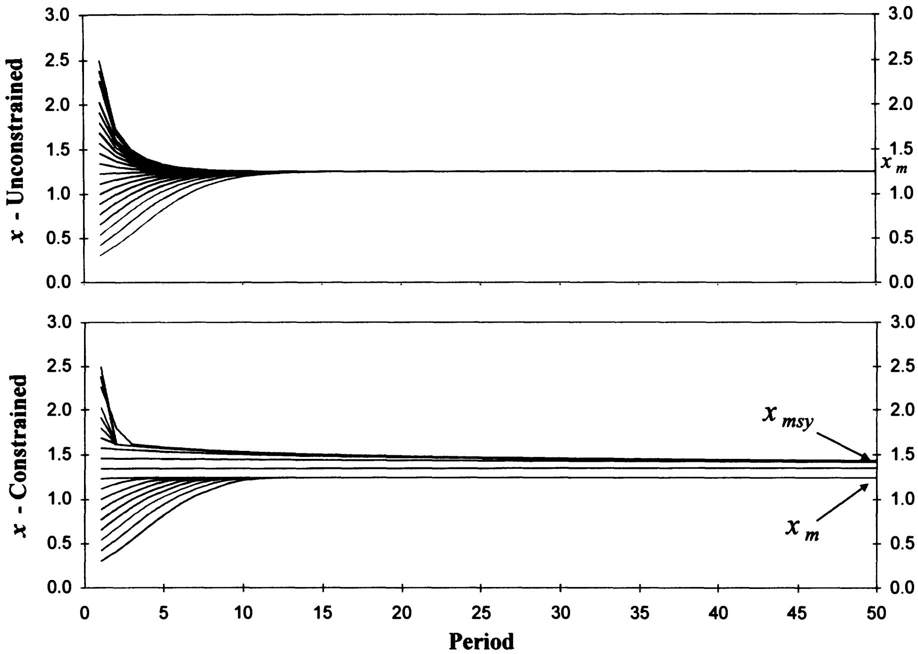

Figure 1. Evolution of the resource endowment in the unconstrained and sustainability-constrained one-dimensional resource economy; parameters of the model: $\beta = 0.9$, $\rho = 0.8$, $\bar{x} = 2$, $\gamma = 0.9$; parameters of the numerical program[13]: order of the Chebyshev polynomial, 30; bounds on state space, 0.3 and 2.5; convergence criterion 10^{-8}

Optimal-Sustainable Management in a One-Dimensional Resource Economy

We now demonstrate the properties of sustainability-constrained economies in one- and two-dimensional economies. We begin with an economy dependent upon a single renewable resource x. In each period the decision maker chooses z_t, the portion of the available stock to consume immediately. The remaining stock, $(1 - z_t)(x_t + \varepsilon_t) = \tilde{x}_t$, grows according to a logistic growth function $x_{t+1} = \tilde{x}_t + \rho\tilde{x}_t(1 - \tilde{x}_t/\bar{x})$ where ρ is the growth rate, $\bar{x}$ is the unexploited steady state, and ε_t is a normally distributed i.i.d. shock with mean zero and standard deviation σ.[12] The portion of the stock that is consumed in t yields utility $u(z_t, x_t, \varepsilon_t) = (z_t(x_t + \varepsilon_t))^{1-\gamma}$ with $\gamma < 1$.

We consider first the case of a policy maker in a deterministic economy ($\sigma = 0$). The top graph in Figure 1 presents the time-path of the resource that would follow from the maximization of each generation's welfare without imposing a sustainability constraint (which we call the PV-optimal policies). As expected (Clark), the PV-optimal trajectory converges to a steady-state level, in this case $x_m = 1.253$, less than the level at which the maximum sustainable yield is achieved, $x_{msy} = 1.40$.

Some of the implications of imposing a sustainability constraint can be anticipated immediately and are equivalent to Calvo's results. Because the unconstrained value function is monotonically increasing in x, at any point from which the PV-optimal policy leads to growth in the resource stock over time, the sustainability constraint would not bind. Over this range, therefore, the PV-optimal and sustainability-constrained optimal (S-optimal) policies would coincide. Sustainability and PV-optimality only diverge, therefore, for initial stocks above x_m.

The bottom graph in Figure 1 presents the paths for this economy that follow from applying the S-optimal policy rule. As anticipated, paths that begin below x_m are identical to those in Figure 1. For initial stocks between x_m and x_{msy}, the S-optimal policy leads to the exact maintenance of that stock level, coinciding with Calvo's maximin criterion. Interestingly, if the initial stock is above x_{msy}, then the S-optimal

[12] For this model, the normal distribution is approximated using a third order Gaussian quadrature approximation using values from Miranda.

[13] Woodward details the numerical methods used in this analysis. This is also available from the author via the internet at http://ageco.tamu.edu/faculty/woodward/.

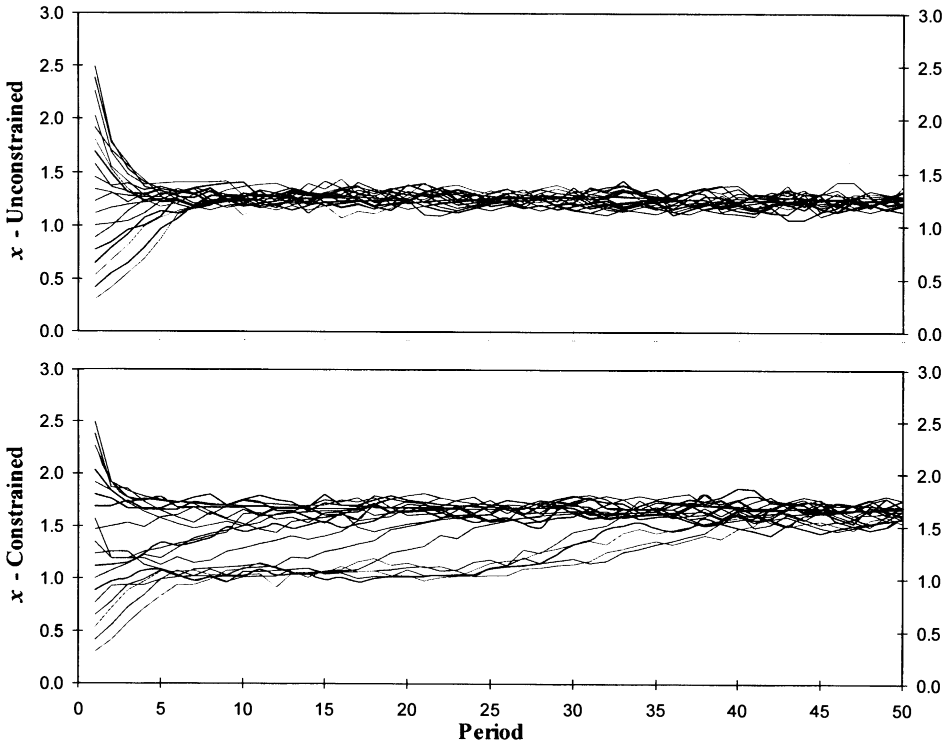

Figure 2. Selected paths of the resource stock value in the unconstrained and sustainability-constrained one-dimensional resource economy under risk ($\sigma = 0.05$) (for other parameter values see Figure 1)

policy rule leads to a gradual decline in the resource stock toward x_{msy}. By definition, no path can lead to a constant stream of harvests that is higher than the maximum sustainable yield. Hence, $V^S(x)$ must reach a maximum at x_{msy}. If the initial resource stock is above x_{msy}, therefore, the S-optimal policy cannot lead to an initial level of utility levels greater than the maximum sustainable level. Hence, if $x_0 > x_{msy}$, then the only policy consistent with the sustainability constraint is to waste the excess stock. This obvious inefficiency is confirmed in the value of λ which reaches 1.0 when $x > x_{msy}$.

Figure 2 presents the PV- and S-optimal paths for the resource-based economy under risk ($\sigma = 0.05$). The PV-optimal paths follow a pattern similar to that observed in Figure 1 but with noise due to the shocks. The addition of risk to the constrained model, however, leads to an important difference in the S-optimal path. Under certainty, S-optimal management of the resource led to a range of steady-states from x_m to x_{msy}. However, when risk is introduced in the constrained case, there is a gradual upward trend in the resource stock over time. In period 10, for example, the average resource stock across the twenty paths presented is 1.41 and by the end of the fifty-period simulation this average has increased to 1.64.

The gradual increase in the stock under risk confirms the theoretical expectation of growth in sustainability-constrained problems. It also suggests, however, that the presence of risk increases the likelihood that the sustainability constraint can lead to inefficiencies. We have seen that if the stock is greater than x_{msy} there is a conflict between efficiency and sustainability. The S-optimal policy under risk leads the economy toward this point.

Optimal-Sustainable Management of a Two-Dimensional Resource Economy

We now consider a two-dimensional capital-resource economy in which, in addition to the renewable resource now identified as x_1, the economy is also dependent upon a reproducible capital stock x_2.[14] Using a

[14] Only the deterministic case is presented here as results from a stochastic model added few insights. The results from a stochastic model can be obtained from the author.

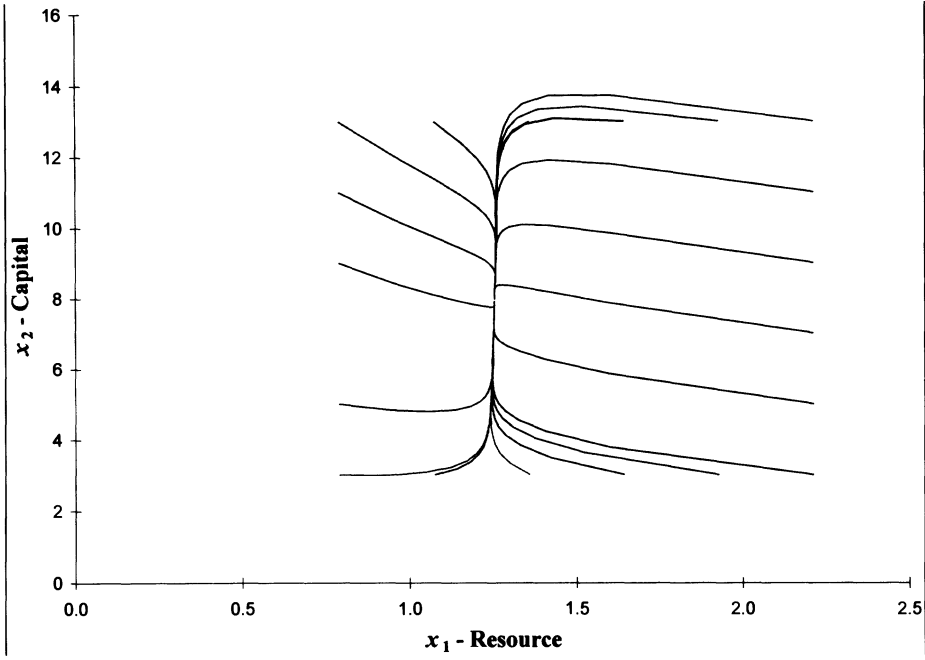

Figure 3. Selected paths of the capital-resource economy in the unconstrained economy; parameters used: the order of the Chebyshev polynomial is 20 in each dimension; the bounds are 0.5 and 2.5 and on x_1 and 1.0 and 15.0 on x_2; and $\alpha = 0.5$ (for all other parameters see figure 1)

Cobb–Douglas production function, the existing capital stock and withdrawals from the resource stock, $z_1 \cdot x_1$, together produce a fungible output. This output can then either be consumed, say c, or invested in the capital stock for the next period, z_2, so that $c + z_2 = (z_1 \cdot x_1)^{\alpha}(x_2)^{1-\alpha}$ and $x_{2t+1} = x_{2t} + z_{2t}$. Again utility is a concave function of consumption, $u(c) = c^{1-\gamma}$.

In the absence of the sustainability constraint, the optimal trajectories lead to a single steady-state equilibrium as seen in Figure 3. For the parameter values used, the steady state occurs at about $x_1 = x_m = 1.25$ and $x_2 = 7.90$.[15] The value function for this model is monotonically increasing in both x_1 and x_2. Hence, paths that increase both assets are consistent with sustainability while paths that reduce both assets violate sustainability. Paths beginning in the northwest or southeast corners lead to increments in one asset and reductions in the other. It is along these paths that the question of sustainability is most interesting.

The S-optimal trajectories in the two-dimensional economy are displayed in Figure 4. For any path beginning in the state space below the dotted line, the sustainability constraint does not bind and the PV-optimal and S-optimal paths coincide. When starting above the dotted line, any reduction in one asset must be compensated by increments to the other. Because of the opportunities for substitution, the effect of the constraint on the management is slight when compared with the one-dimensional case. All the S-optimal paths lead to equilibrium stocks of x_1 near x_m, increasing only slightly for higher initial capital stocks. Because of the ability to substitute capital for the natural resource, the S-optimal use of the natural resource is much more efficient in this model. The constraint does, however, substantially alter the management of x_2. The capital consumption that is seen in Figure 3 is disallowed in the sustainability-constrained model. In effect, therefore, the sustainability constraint becomes a constraint on capital stock, rather than a constraint on the resource.

Because current consumption can always be converted into a greater capital stock, there are always sustainably productive alternatives in this economy. Hence, we know from Proposition 3 that the solution to the problem will be stepwise Pareto efficient. This

[15] While the specific results are sensitive to parameter values chosen, the trends exhibited in the figures are insensitive to parameter choice over a wide range.

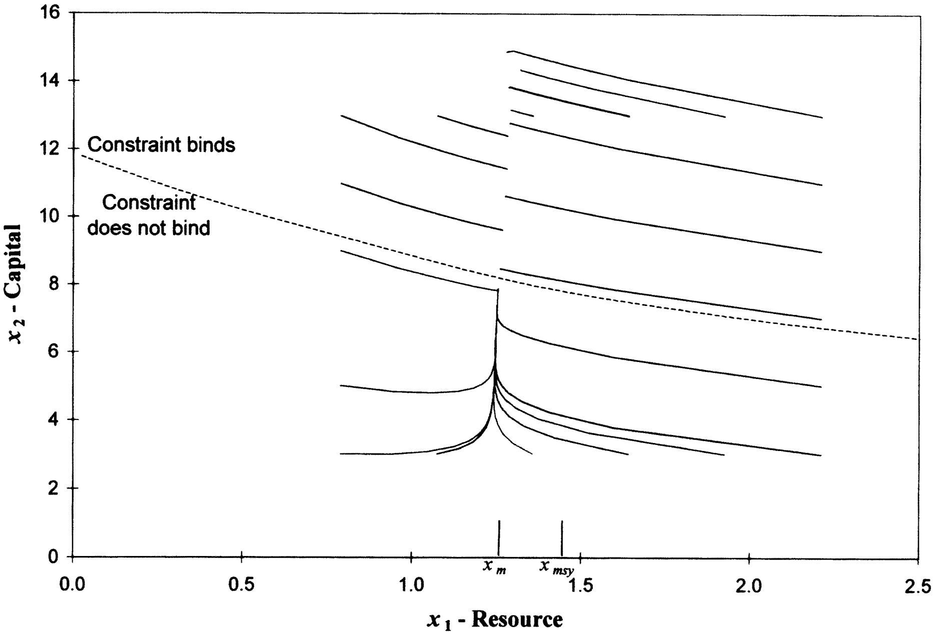

Figure 4. Selected paths of the capital-resource economy in the sustainability-constrained economy (for parameter values see figure 3)

is reflected in the Lagrange multiplier on the sustainability constraint, which reaches a maximum of only 0.25 in the state-space considered.

Conclusions

Sustainability continues to be prominent in public debates about economic development and natural-resource management. Economics has much to offer to these debates. The discipline can offer clarity as to its meaning, explore the implications of commitments to sustainability, and provide guidance as to how sustainability can be achieved. This paper contributes in each of these areas.

First, the principal motivation behind concerns about sustainability is a concern that current policies are intergenerationally unfair. We posit, therefore, that the meaning of sustainability can be clarified using Foley's economic definition of fairness. Assuming policy makers hold intergenerational fairness to be a generational obligation, we arrive at our framework of sustainability-constrained optimization. We find two advantages to this model. First, its recursive framework is a intuitively attractive mathematical formalization of arguments made by others that there is an obligation to treat future generations fairly. Second, it has a practical advantage in that it can be solved numerically by adapting common tools of dynamic programming.

Two theoretical implications of sustainability are discussed. We find that under risk, planning for sustainability requires the expectation of growth. This means that policies in pursuit of sustainability must actually seek sustainable growth. Second, we consider the possibility that the norm of sustainability might conflict with the pursuit of economic efficiency. While we find that the sustainability constraint can lead to Pareto inefficient outcomes, we also show that such inefficiencies are avoided in productive economies. Hence, we are confident that in most situations policy makers can pursue the goal of sustainability without conflicting with the goal of efficiency.

Finally, we provide the groundwork for models that might advance the search for sustainable policies in more complex economies. The numerical methods introduced here vastly expand the types of problems that can be considered in analysis of sustainability. Clearly, the range of problems that might be considered using these methods is still limited, but it is enormous compared to those for which analytical solutions can be obtained.

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Appendix: Propositions and Proofs

Proof that the Successive Approximation Algorithm Will Solve (4)

PROPOSITION 4. *Under Assumptions* B *and* C, *the iterative solution of* (5) *with* $V^0(\mathbf{x}) = V^*(\mathbf{x})$ *converges to* $V^S(\mathbf{x})$ *as* $k \to \infty$.

The final proof is provided below. Following Bertsekas, if V^k is the solution to (5), we can write $V^k = T(V^{k-1})$, where T, the sustainability-constrained Bellman's operator, is defined by (5) and maps from the function V^{k-1} onto the function V^k. This operator has an important monotonicity property.

LEMMA 1. *For any bounded functions* $V^A\colon X \to \mathbb{R}^1$, *and* $V^B\colon X \to \mathbb{R}^1$, *such that* $V^B(\mathbf{x}) \geq V^A(\mathbf{x})$ *for all* $\mathbf{x} \in X$, $spT(V^B(\mathbf{x})) \geq T(V^A(\mathbf{x}))$ *for all* $\mathbf{x} \in X$.

Proof. For a particular point $\mathbf{x}_t$, let Problem A be the maximization of $Eu(\mathbf{z}, \mathbf{x}_t, \varepsilon) + \beta EV^A(g(\mathbf{z}, \mathbf{x}_t, \varepsilon))$ subject to the associated feasibility and sustainability constraints, and let Problem B be the similar problem with V^B substituted for V^A. Suppose $\mathbf{z}^A$ solves Problem A and $\mathbf{z}^B$ solves Problem B. Since $V^B(\mathbf{x}) \geq V^A(\mathbf{x})$, for all $\mathbf{x}$ and $\mathbf{z}^A$ satisfies the sustainability constraint, it follows that $Eu(\mathbf{z}^A, \mathbf{x}_t, \varepsilon) + \beta EV^A(g(\mathbf{z}^A, \mathbf{x}_t, \varepsilon)) \leq Eu(\mathbf{z}^A, \mathbf{x}_t, \varepsilon) + \beta EV^B(g(\mathbf{z}^A, \mathbf{x}_t, \varepsilon))$ and, for $\beta < 1$, that $Eu(\mathbf{z}^A, \mathbf{x}_t, \varepsilon) \leq (1-\beta)EV^A(g(\mathbf{z}^A, \mathbf{x}_t, \varepsilon)) \leq (1-\beta)EV^B(g(\mathbf{z}^A, \mathbf{x}_t, \varepsilon))$. Hence, $\mathbf{z}^A$ is also a feasible solution to Problem B. However, since $\mathbf{z}^B$ solves Problem B,

$$Eu(\mathbf{z}^B, \mathbf{x}_t, \varepsilon) + \beta EV^B(g(\mathbf{z}^B, \mathbf{x}_t, \varepsilon)) \geq Eu(\mathbf{z}^A, \mathbf{x}_t, \varepsilon) + \beta EV^A(g(\mathbf{z}^A, \mathbf{x}_t, \varepsilon)).$$

LEMMA 2. *If* $V^0(\cdot) = V^*(\cdot)$, *then* $V^{k+1}(\mathbf{x}) \leq V^k(\mathbf{x})$ *for all* $x \in X$, $k = 0, 1, \ldots$.

Proof. Let $T^*(V)$ be the standard Bellman's operator (Bertsekas). V^* is defined by the functional fixed point where $T^*(V^*) = V^*$ (Bertsekas). Because of the additional constraint in T, $T(V^*) \leq V^*$. By Lemma 1, $T^2(V^*) \leq T(V^*)$ and, by induction, $T^{k+1}(V) \leq T^k(V)$ for all k.

For purposes of discussion, define $V^\infty \equiv \lim_{k\to\infty} T^k(V^*)$. We now prove the existence and uniqueness of V^∞ and then show that this limit is equal to V^S, the solution to (4).

LEMMA 3. *If Assumptions* B *and* C *are held, then* V^∞ *is bounded from above by* V^* *and from below by* $\underline{u}/(1-\beta)$.

Proof. The upper bound on V^∞ follows immediately from Lemma 2. The lower bound on V^∞ is guaranteed by Assumption B. If $u(\underline{\mathbf{z}}, \mathbf{x}, \varepsilon) = \underline{u}$ for all $\mathbf{x}$ and ε, then the worst possible stream of utility has a present value of stream of $\sum_{t=0}^{\infty} \beta^t \underline{u} = \underline{u}/(1-\beta) \equiv \underline{V}$. Furthermore, since $\underline{u} = (1-\beta)\underline{V} \leq (1-\beta)EV^{k-1}(g(\mathbf{x}_t, \underline{\mathbf{z}}, \varepsilon_t))$, $\underline{\mathbf{z}}$ satisfies the sustainability constraint.

PROPOSITION 4a. *If Assumptions* B *and* C *hold, then* $V^\infty = \lim_{k\to\infty} T^k(V^*)$ *exists.*

Proof. By Lemma 3, for all k, $T^k(V^*) \in [\underline{V}, T^{k-1}(V^*)]$. By Lemma 2, at the kth stage of the algorithm, for all $\mathbf{x} \in X$ there are two possible outcomes, either $T^{k+1}(V^*(\mathbf{x})) < T^k(V^*(\mathbf{x}))$ or $T^{k+1}(V^*(\mathbf{x})) = T^k(V^*(\mathbf{x}))$. If the former holds for all k, then the algorithm will converge to the lower bound $\underline{V}$. If the latter holds for some k, then $T^{k+j}(V^*) = T^k(V^*)$, for all $j = 1, 2, \ldots$.

PROPOSITION 4b. *If Assumptions* B *and* C *hold, then* $V^\infty(\mathbf{x})$ *is equal to* $V^S(\mathbf{x})$.

Proof. If $V^S(\mathbf{x}) < V^\infty(\mathbf{x})$, for a given $\mathbf{x} \in X$, then it is possible that there exists a policy that satisfies the constraints of (4) and yields a higher value than $V^S(\mathbf{x})$. But this contradicts the fact that V^S is the maximum value. Hence, $V^S \geq V^\infty$. By definition $T^k(V^S) = V^S$ and by Lemma 1, $T^k(V^*) \geq T^k(V^S)$. Hence, for all k, $V^k \geq V^S$ and, accordingly, $V^\infty \geq V^S$.

Proof of Proposition 4. Propositions 4a and 4b imply that Proposition 4 holds.

Proof of Proposition 2, the Relationship between Efficiency and λ

LEMMA 4. *If the sustainability constraint is binding at the optimum of* (4), *then an alternative* *policy,* $\mathbf{z}_t \in \Gamma(\mathbf{x}_t)$, *can be stepwise Pareto superior to the optimal policy,* z_t^S, *only if* $E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) > E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$.

Proof. For generation t, the policy $\mathbf{z}_t$ is preferred to $\mathbf{z}_t^S$ if (7a) and (7b) are satisfied, one with a strict inequality. There are three ways that such an inefficiency might occur:

i. (7a) holds with a strict inequality and (7b) with an equality;
ii. (7a) holds with an equality and (7b) with a strict inequality;
iii. (7a) holds with a strict inequality and (7b) with a strict inequality.

If $E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \leq E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$, then (7a) can only hold with a strict inequality if $E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)) > E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$, ruling out *i*. Since $\mathbf{z}_t^S$ is the solution to (4), $E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t) \leq (1-\beta) E_t V(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$. Suppose that $E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \leq E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$, and that (7b) holds with a strict inequality. Then $E_t u_t(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) < (1-\beta) E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t))$. Hence, $\mathbf{z}_t$ does not violate the sustainability constraint, contradicting the assumption that z_t^S is optimal.

Proof of Proposition 2. Let $\mathcal{L}(\mathbf{z}_t, \lambda)$ be the Lagrangian of (4), i.e.,

$$\begin{aligned}\mathcal{L}(\mathbf{z}_t, \lambda_t) = {} & E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\ & + \beta E_t V^S(\mathbf{x}_{t+1}) \\ & - \lambda[E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) \\ & + (\beta - 1) E_t V^S(\mathbf{x}_{t+1})].\end{aligned}$$

For notational convenience, let $u_t^S = E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$, $V_{t+1}^S = E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$, $u_t = E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)$, and $V_{t+1} = E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t))$, where $\mathbf{z}_t \in \Gamma(\mathbf{x}_t)$. The saddle-point condition implies that $u_t(1-\lambda) + [\beta + \lambda(1-\beta)] V_{t+1} \leq u_t^S(1-\lambda) + [\beta + \lambda(1-\beta)] V_{t+1}^S$. By Lemma 4, we need only consider $\mathbf{z}_t$ such that $u_t > u_t^S$. Introducing such a case we obtain

$$\text{(A1)} \qquad \frac{(V_{t+1} - V_{t+1}^S)}{u_t - u_t^S} \leq \frac{(\lambda - 1)}{[\beta + \lambda(1-\beta)]}.$$

Call the left- and right-hand sides of (A1) LHS and RHS, respectively. If for some $\mathbf{z}_t \in \Gamma(\mathbf{x}_t)$, LHS ≥ 0 then z_t^s is stepwise Pareto inefficient. LHS $\geq 0 \Rightarrow$ RHS ≥ 0, which for $0 < \beta < 1$, holds only if $\lambda \geq 1$.

Similarly, $\lambda < 1 \Rightarrow$ RHS $< 0 \Rightarrow$ LHS < 0 for all $\mathbf{z}_t \in \Gamma(\mathbf{x}_t)$ such that $u_t > u_t^S$. Hence, z_t^S is stepwise Pareto efficient.

Proof of Proposition 3, that Productive Economies Are Stepwise Pareto Efficient

LEMMA 5. *If Assumption D holds and there exists a sustainably productive alternative to the optimal choice* $\mathbf{z}_t^S$ *at* $\mathbf{x}_t$ *then* $V^S(\mathbf{x})$ *is not bounded at* $V^S(\mathbf{x_t})$.

Proof. By assumption $E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)) \geq V^S(\mathbf{x}_t)$. We need only consider the case where $E_t V^S(x_{t+1}) = V^S(\mathbf{x}_t)$. If a sustainably productive alternative to $\mathbf{z}_t^S$ exists, then there is some choice $z_t' \in \Gamma(x_t)$ such that $E_t U({}_{t+j}\mathbf{z}, {}_{t+j}\mathbf{x}, {}_{t+j}\varepsilon)\big|_{z_t} \geq V^S(\mathbf{x}_t) + \delta$ for all j. By Assumption D, all future generations can achieve welfare equal to $V^S(\mathbf{x}_t) + \delta$ so that there exists some $\mathbf{x}$ such that $V^S(\mathbf{x}) > V^S(\mathbf{x}_t)$.

We now prove Proposition 3.

Proof. Suppose not. By Lemma 4, if z_t^S is stepwise inefficient, then there exists some $\mathbf{z}_t$, such that $E_t u(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t) > E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$ and $E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t)) \geq E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$. However, using Lemma 5 and Assumption D, if sustainably productive alternatives exist then there is a choice $\mathbf{z}_t'$, alternative to $\mathbf{z}_t$, such that $E_t u(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t) = E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t)$ and $E_t V^S(g(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t)) > E_t V^S(g(\mathbf{z}_t, \mathbf{x}_t, \varepsilon_t))$. This implies that $E_t u(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t) + \beta E_t V^S(g(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t)) > E_t u(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t) + \beta E_t V^S(g(\mathbf{z}_t^S, \mathbf{x}_t, \varepsilon_t))$ and $E_t u(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t) + (\beta - 1) E_t V^S(g(\mathbf{z}_t', \mathbf{x}_t, \varepsilon_t)) \geq 0$, so that $\mathbf{z}_t^S$ cannot solve (4).